

On rough asymptotic behaviour of ruin probabilities in a general discrete risk model *

Jaakko Lehtomaa[†]

March 5, 2013

Abstract

We study the rough asymptotic behaviour of a general economic risk model in a discrete setting. Both financial and insurance risks are taken into account. Loss of the year n is modelled as a random variable $B_1 + A_1 B_2 + \dots + A_1 \dots A_{n-1} B_n$, where A_i corresponds to the financial risk of the year i and B_i represents the insurance risk respectively. Risks of the same year i are not assumed to be independent.

The main result shows that ruin probabilities exhibit power law decay under general assumptions. Our objective is to give a complete characterisation of the relevant quantities that describe the speed at which the ruin probability vanishes as the amount of initial capital grows. These quantities can be expressed as maximal moments, called moment indices, of suitable random variables. In addition to the study of ultimate ruin, the case of finite time interval ruin is considered. Both of these investigations make extensive use of the new properties of moment indices developed during the first half of the paper.

Keywords: Insurance mathematics; Ruin theory; Moment index; Perpetuity; Heavy-tailed

MSC2010: 60E05; 60G07; 60K25; 60K35

1 Introduction

1.1 Motivation

Assume (Ω, \mathcal{F}, P) is a probability space. Define a stochastic process $(Y_n)_{n=1}^\infty = (Y_n)$ on (Ω, \mathcal{F}, P) by formula

$$(1.1) \quad Y_n = \sum_{i=1}^n A_1 \dots A_{i-1} B_i.$$

*This work was financially supported by the Finnish doctoral programme in stochastics and statistics (FDPSS) and by Academy of Finland, grant number 251170.

[†]e-mail address: jaakko.lehtomaa@helsinki.fi

The process (1.1) allows us to define two maximal random variables. We define

$$(1.2) \quad \bar{Y} = \sup_{k \in \mathbb{N}} \{Y_k\}$$

for unbounded time interval and

$$(1.3) \quad \bar{Y}_n = \sup_{1 \leq k \leq n} \{Y_k\}$$

for bounded time interval. Under additional assumptions we may also define perpetuity

$$(1.4) \quad Y_\infty = \sum_{i=1}^{\infty} A_1 \dots A_{i-1} B_i.$$

Processes of the form (1.1) have attracted a fair amount of interest in recent years. Intriguing article [1] investigated this kind of process and recovered results concerning the maximal moments of the random variable $|Y_\infty|$. To name a few others, [21], [22] and [23] have also employed the model (1.1) to figure out the behaviour of the process on a bounded interval. For example, [21] holds results concerning the moments of (1.3) under independence assumption.

The contribution of the present paper is twofold. Firstly, we want to expand the study of moments of absolute values to the study of moments of, say, positive parts of random variables. This allows us to study one-sided level crossing situations that are typically encountered in insurance mathematics in the form of ruin problems. Our proofs do not need complicated mathematical machinery, which allows us to work with truly minimal assumptions. Secondly, the dependence of the financial and insurance risks of the same year has not, to our knowledge, been investigated in the context of determining the maximal moments of processes (1.2) and (1.3). Yet, such dependence can be found in many applications.

To give an example, we recall the famous ARCH(1) process (X_n) with parameters $\beta, \lambda > 0$. It is defined recursively by formula

$$X_n = \sqrt{\beta + \lambda X_{n-1}^2} Z_n, \quad n \in \mathbb{N},$$

where (Z_n) is a sequence of independent and identically distributed, abbreviated IID, random variables and X_0 is independent of (Z_n) . The squared process (X_n^2) may be written in form

$$(1.5) \quad X_n^2 = \beta Z_n^2 + \lambda Z_n^2 X_{n-1}^2.$$

If $X_0 = 0$, iteration of (1.5) yields a random variable whose distribution is that of (1.1) with $(A_i, B_i) = (\lambda Z_i^2, \beta Z_i^2)$. Clearly, A_i and B_i are not independent in this case. In chapter 2.1 below the connection between (1.1) and recursive equations of the type (1.5) is explained in detail.

Another natural example of the occurrence of (1.1) with dependent risks can be found from the continuous time analogue of (1.1). Using the notation of Nyrhinen in Section 3 of [16], the continuous time analogue $(Y_t^c)_{t \geq 0}$ of the process (Y_n) is defined by

$$(1.6) \quad Y_t^c = \int_0^t A_{s-}^c dB_t^c.$$

In (1.6) the process $((\log A_t^c, B_t^c))_{t \geq 0}$ is defined on a suitable filtered probability space $(\Omega, \mathcal{F}_{t \geq 0}, \mathbb{P})$. It is assumed to be stationary with independent increments and càdlàg sample paths. With choices

$$A_t^c = \prod_{m=1}^{\lfloor t \rfloor} A_m$$

and

$$B_t^c = \sum_{m=1}^{\lfloor t \rfloor} B_m$$

model (1.1) can be recovered recovered from (1.6). By terminology of [16], *the associated discrete time process* of (1.6) is defined by fixing the IID random vectors $(A_1, B_1), (A_2, B_2), \dots$ via equation

$$(1.7) \quad (A, B) \stackrel{d}{=} (A_1^c, Y_1^c),$$

where $\stackrel{d}{=}$ denotes equality in distribution. The interesting observation is that the vector (A, B) does not necessarily have independent components even if the original processes (A_t^c) and (B_t^c) are independent. For further background to processes of the type (1.6) and ruin analysis in these models can be found from the work of Paulsen in [19]. The reader is also advised to see the survey article [18] from the same author.

After all, these observations indicate that dependence between financial and insurance risks is natural and should be included in the risk model. In addition, since ruin problems of continuous processes can be often reduced to the analysis of the associated discrete time process, it is beneficial to understand the behaviour of these simpler discrete processes in detail.

1.2 Assumptions

The process (Y_n) defined in (1.1) describes the total loss of an insurer at the end of the year $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. We assume that the sequence $(A_1, B_1), (A_2, B_2), \dots$ satisfies the following assumptions.

- I The sequence $((A_i, B_i))$ consists of IID random vectors.
- II The members of (A_i) are strictly positive and A is not the constant 1.

III The members of (B_i) are real valued and $P(B > 0) > 0$.

In a sense, assumptions II and III are minimal. If A was the constant 1, model (1.1) would reduce to a classical random walk. However, random walks have entirely different asymptotic behaviour. This is the reason why they are only mentioned in comments section 7. On the other hand, if equality $P(B > 0) = 0$ was valid, ruin would be impossible.

1.3 Background

Assume the insurer operates with initial capital $U_0 > 0$. Denote by U_n the capital of the insurer at the end of the year n . Let r_n be the associated stochastic return on investment over the period from $n - 1$ to n . Our assumption is that the sequence (U_n) satisfies the recursive formula

$$(1.8) \quad U_n = (1 + r_n)(U_{n-1} - B_n).$$

For further background of this recursion and the relation between (1.1) and (1.8), see Nyrhinen [15],[16] and especially [17].

The random variable A represents the financial risk while B corresponds to the insurance risk. The random variable A can be regarded as a stochastic discounting factor $1/(1 + r)$. Since $A > 0$, the random variable r must satisfy inequality $r > -1$. This means that losses on financial markets are possible, but one cannot lose more than what is initially invested.

We define the time of ruin $T_{U_0} = T$ by

$$(1.9) \quad T = \inf\{n : U_n < 0\}$$

with the use of convention $\inf\{\emptyset\} = \infty$. Formula (1.9) can be rewritten in alternative form

$$(1.10) \quad T = \inf\{n : Y_n > U_0\}$$

as is proven in [17]. These two representations of ruin time act as a link between the processes (U_n) and (Y_n) . Clearly, $\{T < \infty\} = \{\bar{Y} > U_0\}$ and $\{T \leq n\} = \{\bar{Y}_n > U_0\}$.

Next we recall some general notations. Let x^+ denote the positive part $\max(0, x)$ of a real number x . If X is a random variable, the quantity $\mathbb{I}(X) \in [0, \infty]$ is defined by formula

$$(1.11) \quad \mathbb{I}(X) = \sup\{s \geq 0 : E((X^+)^s) < \infty\}.$$

Quantity $\mathbb{I}(X)$ is called *the moment index* of the random variable X . Equation (1.11) is reasonable because the function $E((X^+)^s)$ is a convex function on \mathbb{R} . For a comprehensive treatment of convex functions, see [20].

We say that a random variable X is (right) *heavy-tailed* if no exponential moments exist. That is, if $\sup\{s \geq 0 : E(e^{sX}) < \infty\} = 0$. We note that the moment

index of a random variable may obtain finite value only when X is heavy tailed. In fact, moment index can be seen as a measure of risk of heavy tailed distributions.

As a general rule one may argue that a smaller moment index means larger risk in the sense of a heavier tail. Heavy tail on the other hand implies the increased possibility of very large realisations. From this fact we see that the moment index is important in questions related to ruin. For an introduction to heavy-tailed and especially subexponential distributions, the reader is referred to look at [11]. Furthermore, to see how such distributions affect the behaviour of random walks, see [5].

It is known from Daley [8] Lemma BDK or [2] that there is a connection between the moment index and the tail of a random variable. Namely,

$$(1.12) \quad \limsup_{x \rightarrow \infty} \frac{\log \bar{F}_X(x)}{\log x} = -\mathbb{I}(X),$$

where the function \bar{F}_X is the tail function $1 - F_X$ of random variable X and F_X is the distribution function of X . Here convention $\log 0 = -\infty$ is used. Using the terminology of [3], $-\mathbb{I}(X)$ corresponds to the upper order of the function \bar{F}_X .

Another quantity, which is closely related to the moment index, can be defined for a positive random variable Y by formula

$$(1.13) \quad \mathbb{I}^1(Y) = \sup\{s \geq 0 : E(Y^s) \leq 1\}.$$

We note that inequality $\mathbb{I}^1(Y) \leq \mathbb{I}(Y)$ holds between (1.11) and (1.13). Quantities similar to (1.13) are important in classical ruin theory as well as in theory of large deviations. For a thorough treatment of the theory of large deviations the reader is advised to see [9]. In chapter 7.2 below the quantity $\mathbb{I}^1(Y)$ is characterised via moment indices.

1.4 The aim of the study

Our main goal is to discover the value of

$$(1.14) \quad \mathbb{I}(\bar{Y}) = -\limsup_{U_0 \rightarrow \infty} \frac{\log P(T < \infty)}{\log U_0}$$

using as few assumptions as possible. The range in which

$$(1.15) \quad \mathbb{I}(\bar{Y}_n) = -\limsup_{U_0 \rightarrow \infty} \frac{\log P(T \leq n)}{\log U_0}$$

lies will also be discovered.

The quantities (1.14) and (1.15) describe the rough asymptotic behaviour of the probabilities $P(T < \infty)$ and $P(T \leq n)$. Unfortunately, in general, there are no guarantees that the limes superiors of (1.14) and (1.15) coincide with corresponding limes inferiors. However, from the viewpoint of insurance, the upper estimate obtained from these limes superiors is actually the more important one.

Suppose the limes superior of (1.14) satisfies $\limsup \log P(T < \infty) / \log U_0 = -\alpha$, where $\alpha \in (0, \infty)$. Now, we immediately see that for any $\varepsilon > 0$ the estimate $P(T > \infty) \leq U_0^{-\alpha+\varepsilon}$ holds for all U_0 large enough. Since upper bounds for ruin probabilities are the most important, limes superior offers the sufficient knowledge for this application.

We will show which quantities determine the value of α . It is natural to think that the risk determining the value of (1.14) is the dominating risk. In this sense our study tells which of the risks the dominating one is. The value of α related to the unbounded interval is determined by the smaller of the quantities $\mathbb{I}^1(A)$ and $\mathbb{I}(B)$. Our approach utilises monotony properties of the model. This enables us to cut ourselves loose from the usual, yet unneeded, technical assumptions such as $\mathbb{I}^1(A) < \mathbb{I}(A)$ or $\mathbb{I}^1(A) < \mathbb{I}(B)$.

The structure of the article is built to support the main results presented in sections 3-6. Section 2 includes the necessary preliminaries and develops the theory of moment indices. Section 3 is dedicated to the different estimation methods that follow from the monotonicity of the model. Section 4 clarifies the assumptions needed in the proof of main theorem presented in sections 5 and 6. Finally, section 7 contains comments and conclusions.

1.5 Related studies

The closest study to our purpose is Nyrhinen [16], where a similar problem is partially considered in Theorem 2. The present paper proves that the result concerning the limit supremum of ultimate ruin holds in a more general setting. In addition, required assumptions are simplified.

There exists a multitude of papers that study the model (1.1) when further assumptions on the distributions of A and B are made. Studies of Tang and Tsitsashvili [21] and [22] use the model (1.1). However, these papers concentrate on sharp asymptotic results in a model where more detailed assumptions are made on the distributions of random variables.

The connection of limiting distributions and stochastic fixed point equations is used in most of the cited articles. However, to our knowledge these equations have not been combined with moment indices. A recent article [17] considers stochastic equations satisfied by the random variables \tilde{Y} and \tilde{Y}_n . In this paper we will use these connections to find a representation for $\mathbb{I}(\tilde{Y})$ and $\mathbb{I}(\tilde{Y}_n)$.

Our result is about the the maximal moment of random variable \tilde{Y} . Similar viewpoint has been studied in [1] where the absolute moments of process (1.4) are explored. The article [1] raises a question on the moment properties of \tilde{Y} . In present paper this question is answered in the case of heavy tailed risks.

It is possible to use the connection

$$(1.16) \quad P(\tilde{Y} > u) \geq P(Y_\infty > u) \geq P(Y_\infty > 0)P(\tilde{Y} > u)$$

mentioned in remark 2.1 of [17] to see that $\mathbb{I}(\tilde{Y}) = \mathbb{I}(Y_\infty)$, when the series Y_∞ converges and $P(Y_\infty > 0) > 0$. Exact conditions that ensure the almost sure existence

of the limit variable (1.4) can be found from [12] and [13]. For convergence in IID-case it suffices that $E(\log^+ |B|) < \infty$ and $E(\log A) < 0$ by [10] chapter 8.4.

Outside the field of insurance, similar problems are encountered in statistical ARCH and GARCH models, as briefly mentioned in section 1.1, as well as in queuing theory. Viewpoints concerning this relation can be found in [6], where the dependence assumptions of the process are further relaxed by introducing a Markovian dependence structure and using techniques developed in [14].

2 Tools for analysis

2.1 Stochastic equations

The process (1.1) has many comfortable properties in terms of distributional equations. More precisely, random variables (1.2) and (1.3) satisfy useful random difference and fixed point equations.

Consider a random variable defined at discrete time points n recursively by formula

$$(2.1) \quad V_0 = 0, \quad V_n \stackrel{d}{=} B_n + A_n V_{n-1}^+, \quad n \geq 1,$$

where the random vector (A_n, B_n) is independent of V_{n-1} . Since

$$\begin{aligned} \bar{Y}_n &= \max(B_1, B_1 + A_1 B_2, \dots, B_1 + A_1 B_2 + \dots + A_1 \dots A_{n-1} B_n) \\ &= B_1 + A_1 \max(0, B_2, \dots, B_2 + A_2 B_3 + \dots + A_1 \dots A_{n-1} B_n), \end{aligned}$$

it is easy to see that $\bar{Y}_n \stackrel{d}{=} V_n$ for all $n \in \mathbb{N}$. This connection allows us to calculate some of the moment indices recursively.

In following formulas (2.2) and (2.3) the vector (A, B) is independent of \bar{Y} or Y_∞ on the right hand side. It is known from [17] that

$$(2.2) \quad \bar{Y} \stackrel{d}{=} B + A\bar{Y}^+.$$

In addition, the random variable Y_∞ representing the perpetuity satisfies a simple equation

$$(2.3) \quad Y_\infty \stackrel{d}{=} B + AY_\infty.$$

This equation is often useful because under general assumptions equality $\mathbb{I}(Y_\infty) = \mathbb{I}(\bar{Y})$ holds via equation (1.16).

2.2 Properties of moment indices

Apart from [7] and [8], sources for the use and properties of moment indices have been scarce. Hence, we take the opportunity to present some general results.

Much of the following deduction is based on the fact that random variables with the same distributions share a common moment index. This result can be seen directly.

Lemma 1. Let X and Y be random variables such that $X \stackrel{d}{=} Y$. Then

$$\mathbb{I}(X) = \mathbb{I}(Y).$$

□

It is known from [21] that $\mathbb{I}(X + Y) = \mathbb{I}(XY) = \min(\mathbb{I}(X), \mathbb{I}(Y))$, if random variables X and Y are independent and one of them is non-negative. We will develop similar results under more lenient assumptions. Especially assumptions on independence are alleviated. This is important when one considers vectors, whose components are not independent.

We begin with a list of the most basic properties of moment indices.

Lemma 2. Let X and Y be random variables such that $Y \leq X$ almost surely. Assume $a > 0$ and $b \in \mathbb{R}$. Then

1.

$$\mathbb{I}(aX + b) = \mathbb{I}(X)$$

2.

$$\mathbb{I}((X^+)^a) = \frac{\mathbb{I}(X)}{a}$$

3.

$$\mathbb{I}(X) \leq \mathbb{I}(Y)$$

4.

$$\mathbb{I}(X \mathbf{1}(X > b)) = \mathbb{I}(X).$$

Proof. Connection (1.12) yields

$$\begin{aligned} \limsup_{x \rightarrow \infty} \frac{P(aX + b > x)}{\log x} &= \limsup_{x \rightarrow \infty} \frac{P(X > (x - b)/a)}{\log x} \\ &= \limsup_{y \rightarrow \infty} \frac{P(X > y)}{\log(ay + b)} \\ &= \limsup_{y \rightarrow \infty} \frac{\log y}{\log(ay + b)} \frac{P(X > y)}{\log y} \\ &= \limsup_{y \rightarrow \infty} \frac{P(X > y)}{\log y}, \end{aligned}$$

which proves part 1.

Parts 2 and 3 are obvious from the definition of moment index (1.11) and part 4 is clear by (1.12). □

The following lemma presents a general property of moment index.

Lemma 3. Let X and Y be random variables. Then

$$\mathbb{I}(\max(X, Y)) = \min(\mathbb{I}(X), \mathbb{I}(Y)).$$

Proof. Note that for any $s \geq 0$

$$\begin{aligned} & E((\max(X, Y))^s \mathbf{1}(\max(X, Y) > 0)) \\ &= E(X^s \mathbf{1}(\max(X, Y) > 0, X > Y)) + E(Y^s \mathbf{1}(\max(X, Y) > 0, X \leq Y)) \\ &\leq E(X^s \mathbf{1}(X > 0)) + E(Y^s \mathbf{1}(Y > 0)), \end{aligned}$$

which implies $\mathbb{I}(\max(X, Y)) \geq \min(\mathbb{I}(X), \mathbb{I}(Y))$.

For the other direction we use the connection (1.12) and estimate

$$\limsup_{x \rightarrow \infty} \frac{\log P(\max(X, Y) > x)}{\log x} \geq \begin{cases} \limsup_{x \rightarrow \infty} \frac{\log P(X > x)}{\log x} \\ \limsup_{x \rightarrow \infty} \frac{\log P(Y > x)}{\log x}. \end{cases}$$

Hence $-\mathbb{I}(\max(X, Y)) \geq \max(-\mathbb{I}(X), -\mathbb{I}(Y))$. \square

The next lemma states two important rules of moment indices. They are partially known from [21].

Lemma 4. Let X and Y be random variables. Then

$$(2.4) \quad \mathbb{I}(X + Y) \geq \min(\mathbb{I}(X), \mathbb{I}(Y)).$$

Furthermore, if X and Y are independent or non-negative,

$$\mathbb{I}(X + Y) = \min(\mathbb{I}(X), \mathbb{I}(Y)).$$

Proof. If $\min(\mathbb{I}(X), \mathbb{I}(Y)) = 0$ there is nothing to prove in (2.4). Assume that there exists $0 < s < \min(\mathbb{I}(X), \mathbb{I}(Y))$. By Lemma 3 equality $\min(\mathbb{I}(X), \mathbb{I}(Y)) = \mathbb{I}(\max(X, Y))$ holds. Consequently,

$$\begin{aligned} E((X + Y)^s \mathbf{1}(X + Y > 0)) &\leq E((2 \max(X, Y))^s \mathbf{1}(X + Y > 0)) \\ &\leq 2^s E((\max(X, Y))^s \mathbf{1}(\max(X, Y) > 0)) \\ &< \infty. \end{aligned}$$

Hence $\mathbb{I}(X + Y) \geq \mathbb{I}(\max(X, Y))$ and (2.4) holds.

Assume then that random variables X and Y are independent. As the other direction follows directly from equation (2.4) it suffices to prove inequality $\mathbb{I}(X + Y) \leq \min(\mathbb{I}(X), \mathbb{I}(Y))$. To do this, we begin by choosing a real number b such that $P(X > b) > 0$. Now, for any $0 < s < \mathbb{I}(X + Y)$, we get

$$\begin{aligned} E(((X + Y)^+)^s) &\geq E(((X + Y)^+)^s \mathbf{1}(X > b)) \\ &\geq E(((b + Y)^+)^s \mathbf{1}(X > b)) \\ &= E(((b + Y)^+)^s) P(X > b). \end{aligned}$$

By Lemma 2 part 1 $\mathbb{I}(Y + b) = \mathbb{I}(Y)$ holds and we see that

$$\mathbb{I}(X + Y) \leq \mathbb{I}(Y).$$

Using symmetry of random variables X and Y we also obtain

$$\mathbb{I}(X + Y) \leq \mathbb{I}(X).$$

Therefore $\mathbb{I}(X + Y) \leq \min(\mathbb{I}(X), \mathbb{I}(Y))$.

Assume finally that random variables X and Y are non-negative. Now the events $\{X \geq 0\}$, $\{Y \geq 0\}$ and $\{X + Y \geq 0\}$ are almost sure events. Hence

$$\begin{aligned} E((X + Y)^s \mathbf{1}(X + Y > 0)) &= E((X + Y)^s \mathbf{1}(X + Y \geq 0)) \\ &= E((X + Y)^s) \\ &\geq \max(E(X^s), E(Y^s)) \\ &= \max(E(X \mathbf{1}(X > 0))^s, E(Y \mathbf{1}(Y > 0))^s), \end{aligned}$$

which proves the claim. \square

It is useful to notice the following property of moment indices and the related corollary.

Lemma 5. Let X be a random variable on probability space (Ω, \mathcal{F}, P) and $S \in \mathcal{F}$. Then

$$\mathbb{I}(X) = \min(\mathbb{I}(X \mathbf{1}_S), \mathbb{I}(X \mathbf{1}_{S^c})).$$

Proof. Using the decomposition

$$X^s \mathbf{1}(X > 0) = X^s \mathbf{1}(X > 0) \mathbf{1}_S + X^s \mathbf{1}(X > 0) \mathbf{1}_{S^c}$$

and positivity we may estimate

$$(2.5) \quad \max[E(X^s \mathbf{1}(X > 0) \mathbf{1}_S), E(X^s \mathbf{1}(X > 0) \mathbf{1}_{S^c})] \leq E(X^s \mathbf{1}(X > 0)).$$

On the other hand,

$$(2.6) \quad \begin{aligned} E(X^s \mathbf{1}(X > 0)) &= E(X^s \mathbf{1}(X > 0) \mathbf{1}_S + X^s \mathbf{1}(X > 0) \mathbf{1}_{S^c}) \\ &\leq 2E(\max[X^s \mathbf{1}(X > 0) \mathbf{1}_S, X^s \mathbf{1}(X > 0) \mathbf{1}_{S^c}]). \end{aligned}$$

By equation (2.5) we see that $\mathbb{I}(X) \leq \min(\mathbb{I}(X \mathbf{1}_S), \mathbb{I}(X \mathbf{1}_{S^c}))$. From equation (2.6) we get

$$(2.7) \quad \mathbb{I}(X) \geq \mathbb{I}(\max[X^s \mathbf{1}(X > 0) \mathbf{1}_S, X^s \mathbf{1}(X > 0) \mathbf{1}_{S^c}]).$$

Applying Lemma 3 to the right hand side of formula (2.7) ends the proof. \square

As a direct consequence of Lemma 5 we obtain the following corollary.

Corollary 1. *Let Θ be a finite cover of the state space Ω . Then by Lemma 5*

$$\mathbb{I}(X) = \min_{\theta \in \Theta} \mathbb{I}(X \mathbf{1}_\theta).$$

The Corollary 1 motivates to define a new conditional quatity. If X is a random variable and H an event of the state space Ω with $P(H) > 0$, we define *the conditional moment index* $\mathbb{I}(X|H)$ of random variable X by setting

$$\mathbb{I}(X|H) = \sup\{s \geq 0 : E((X^+)^s|H) < \infty\} = \sup\{s \geq 0 : E((X^+)^s \mathbf{1}_H) < \infty\}.$$

We end the section with an example that shows how the stochastic fixed point equations can be used with moment indices.

Example 1. Consider the model (1.1) satisfying assumptions I-III of section 1.2. Assume further that the sequence (B_i) consists of non-negative random variables. Then we may deduce an upper bound for the moment index of random variable \bar{Y} . Namely,

$$(2.8) \quad \mathbb{I}(\bar{Y}) \leq \min(\mathbb{I}(A), \mathbb{I}(B)).$$

Equation (2.8) is a direct consequence of the stochastic equation the random variable \bar{Y} satisfies. We recall from (2.2) that $\bar{Y} \stackrel{d}{=} B + A\bar{Y}^+$. By lemmas 1 and 4 we get

$$(2.9) \quad \mathbb{I}(\bar{Y}) = \min(\mathbb{I}(A), \mathbb{I}(B), \mathbb{I}(\bar{Y})).$$

Obviously (2.9) is equivalent to (2.8).

Two remarks are in order. Firstly, a similar bound could be obtained along the same line of thought if we assumed that the sequence (A_i) was independent of (B_i) . In this case insurance risks could be real valued random variables. Secondly, the bound obtained in (2.8) is not the best possible. In the case $\mathbb{I}^1(A) < \min(\mathbb{I}(A), \mathbb{I}(B))$ the best bound is $\mathbb{I}^1(A)$. This will become clear after the proof of the main theorem in section 6.

3 Estimation methods

We will study how the process (1.1) changes when the original risks are replaced by almost surely smaller risks. The new risks, as well as all quantities related to them, are separated from the original ones by asterisk symbol $*$.

3.1 Monotony of the financial risk

Let us assume that A is replaced by random variable A^* , for which $A \geq A^* > 0$. Assume further that every other aspect of the model remains unchanged. Especially the insurance risk B has the same marginal distribution as before.

We will examine the relationship between the ruin times T and T^* . Recall that the capital at the end of the year n is given by the random variable U_n that satisfies the recursion (1.8). We shall prove the inequality

$$(3.1) \quad U_n \leq U_n^*$$

for every n before the time of ruin, when $A_n \geq A_n^*$, that is, $r_n \leq r_n^*$.

Formula (3.1) can be based on the following argument. We note first that before the time of ruin T each of the random variables $U_{n-1} - B_n$ is positive. If $T = 1$, it follows that $U_1^* \leq U_1 < 0$ and therefore $T^* = T$. Now, if $T \geq 2$, we may use inductive reasoning for each point of the sample space.

1. If $T = k$, where $k \geq 2$, at time $n = 1$ the inequality

$$U_1 = (1 + r_1)(U_0 - B_1) \leq (1 + r_1^*)(U_0 - B_1) = U_1^*,$$

holds and (3.1) is valid.

2. Assume that (3.1) holds for all $1 \leq n \leq T - 2$. Then

$$\begin{aligned} U_{n+1} = (1 + r_{n+1})(U_n - B_{n+1}) &\leq (1 + r_{n+1}^*)(U_n - B_{n+1}) \\ &\leq (1 + r_{n+1}^*)(U_n^* - B_{n+1}). \end{aligned}$$

The order of U and U^* cannot be deduced exactly at the time of ruin. However, it is possible to infer that the process (U_n^*) cannot obtain its first negative value before the process (U_n) has done so. Hence $T^* \geq T$, when $A \geq A^*$.

3.1.1 Case $\mathbb{I}^1(A) = 0$ by estimation of financial risk A

We will show that the random variable A can be estimated from below to overcome the technical problem posed by the pathological situation $\mathbb{I}^1(A) = 0$.

The result $T \leq T^*$ from section 3.1 allows us to deduce

$$P(T_{U_0} < \infty) \geq P(T_{U_0}^* < \infty).$$

This in turn enables the estimate

$$(3.2) \quad \limsup_{U_0 \rightarrow \infty} \frac{\log P(T_{U_0} < \infty)}{\log U_0} \geq \limsup_{U_0 \rightarrow \infty} \frac{\log P(T_{U_0}^* < \infty)}{\log U_0}.$$

The following lemma shows that the needed random variable A^* exists.

Lemma 6. Let $A > 0$ be a random variable such that $\mathbb{I}^1(A) < \infty$. Fix $\varepsilon > 0$. Then there exists a bounded random variable A_ε that fulfils the conditions

1. $A \geq A_\varepsilon > 0$ almost surely and
2. $\mathbb{I}^1(A_\varepsilon) = \mathbb{I}^1(A) + \varepsilon$.

Proof. We begin the construction of the random variable A_ε by fixing three real numbers. Choose $M_1 > 0$ such that

$$E(A^{\mathbb{I}^1(A)+\varepsilon}\mathbf{1}(A \leq M_1)) > 1.$$

Next, choose $0 < M_2 < M_1$ in a way that the inequality

$$E(A^{\mathbb{I}^1(A)+\varepsilon}\mathbf{1}(A \leq M_2)) < 1.$$

holds. Finally, set $0 < M_3 < M_2$ so that

$$E(A^{\mathbb{I}^1(A)+\varepsilon}\mathbf{1}(A \leq M_2) + M_3^{\mathbb{I}^1(A)+\varepsilon}\mathbf{1}(A > M_1)) < 1.$$

We will then estimate the random variable A from below in the set $\{M_2 \leq A \leq M_1\}$ by simple positive functions. We choose the simple functions so that their sum G satisfies the equation

$$E(A^{\mathbb{I}^1(A)+\varepsilon}\mathbf{1}(A \leq M_2) + G^{\mathbb{I}^1(A)+\varepsilon}\mathbf{1}(M_2 \leq A \leq M_1) + M_3^{\mathbb{I}^1(A)+\varepsilon}\mathbf{1}(A > M_1)) = 1.$$

The definition of expectation guarantees the existence of such functions. Figure 1 illustrates the procedure in case $\Omega = \mathbb{R}$. We end the construction by setting

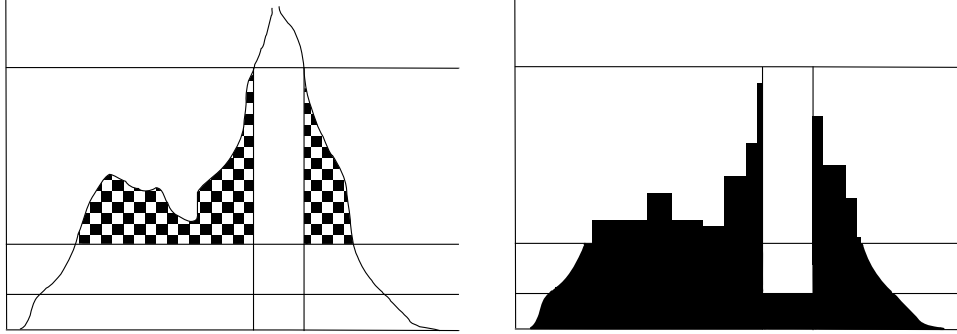


Figure 1: On the left hand side, the graph of a financial risk A on $\Omega = \mathbb{R}$ is shown. The three horizontal lines correspond, from top to bottom, to the levels M_1, M_2 and M_3 respectively. Area filled with chequer board pattern illustrates the region at which the values of the random variable A will be modified by approximation. On the right hand side, the region filled with black presents the graph of random variable A_ε . The part of the graph between levels M_1 and M_2 represents the sum G .

$$(3.3) \quad A_\varepsilon = A^\varepsilon \mathbf{1}(A \leq M_2) + G^\varepsilon \mathbf{1}(M_2 \leq A \leq M_1) + M_3^\varepsilon \mathbf{1}(A > M_1).$$

Now the random variable A_ε defined in (3.3) obviously satisfies conditions 1 and 2. \square

3.2 Monotony of insurance risk

Insurance risks have a similar monotonicity property as financial risks. This property is easy to see, since from $B^* \leq B$ it follows that $Y_k^* \leq Y_k$ for all k . Hence $\bar{Y}^* \leq \bar{Y}$ and $P(T_{U_0} < \infty) \geq P(T_{U_0}^* < \infty)$.

The next result demonstrates how the moment index of a random variable can be increased by estimation from below.

Lemma 7. Let B be a real valued random variable. Assume $\mathbb{I}(B) = \alpha < \infty$ and $\beta > \alpha$.

Then there exists a random variable B^* such that $B^* \leq B$ almost surely and $\mathbb{I}(B^*) = \beta$.

Proof. We will construct the necessary random variable B^* as a minimum of two independent random variables.

Let W be a Pareto distributed random variable independent of B with parameter $\beta - \alpha$. By a Pareto distributed random variable with parameter $\gamma > 0$ we mean a random variable whose tail function is

$$P(W > x) = \begin{cases} \frac{1}{x^\gamma} & : x \geq 1 \\ 1 & : x < 1. \end{cases}$$

Set $B^* = \min(B, W)$. Now

$$\begin{aligned} \limsup_{x \rightarrow \infty} \frac{\log \bar{F}_{B^*}(x)}{\log x} &= \limsup_{x \rightarrow \infty} \frac{\log[P(B > x)P(W > x)]}{\log x} \\ &= \limsup_{x \rightarrow \infty} \left(\frac{\log P(B > x)}{\log x} + \frac{\log P(W > x)}{\log x} \right) \\ &= \limsup_{x \rightarrow \infty} \frac{\log P(B > x)}{\log x} + \lim_{x \rightarrow \infty} \frac{\log P(W > x)}{\log x} \\ &= -\alpha - (\beta - \alpha) = -\beta, \end{aligned}$$

which ends the construction. \square

4 Essential suprema of \bar{Y}_k and \bar{Y}

Let us recall that for a general random variable X the *essential supremum* is defined as

$$\begin{aligned} \text{ess sup } X &= \text{ess sup}_{\omega \in \Omega} X(\omega) = \inf\{a \in \mathbb{R} : P(X > a) = 0\} \\ &= \sup\{a \in \mathbb{R} : P(X > a) > 0\}. \end{aligned}$$

We set $\bar{y} = \text{ess sup } \bar{Y}$ and define in analogous way a number that describes the essential supremum of the random variable \bar{Y}_k . Put $\bar{y}_0 = 0$ and $\bar{y}_k = \text{ess sup } \bar{Y}_k$, when $k \in \mathbb{N}$. Following analysis will reveal how the sequence (\bar{y}_k) behaves.

Dependence structure of the process (Y_k) may lead to unexpected behaviour. Following example demonstrates how the whole process may be limited almost surely for the first $N + 1$ years and then become essentially unbounded.

Example 2. Suppose $N \in \mathbb{N}$ is fixed. Let W_γ , for $\gamma > 0$, be a Pareto distributed random variable with parameter γ . Assume W_α and W_β are independent with $0 < \beta < \alpha$. Assume further that K is independent of W_α and W_β with $P(K = 0) = P(K = 1) = \frac{1}{2}$. We define (A, B) by

$$B = \mathbf{1}(K = 0) - NW_\alpha \mathbf{1}(K = 1)$$

and

$$A = \mathbf{1}(K = 0) + W_\alpha \mathbf{1}(K = 1).$$

Clearly $\bar{y}_k = k$ for all $k = 1, 2, \dots, N+1$. However, $\bar{y}_{N+2} = \infty$.

The moment index of a bounded random variable is infinite. This is why a systematic method of determining the value of \bar{y}_k with respect to a given vector (A, B) is needed. The next theorem summarises how the sequence (\bar{y}_k) behaves and offers a new characterisation for the condition $\bar{y} = \infty$.

Theorem 1. Assume that the IID random vectors (A, B) satisfy

- (i) $P(A > 0) = 1$
- (ii) $P(A > 1) > 0$
- (iii) $P(A < 1) > 0$
- (iv) $P(B > 0) > 0$.

Then, for any $N \in \mathbb{N}$,

$$(4.1) \quad \bar{y}_N = \text{ess sup}(B + \bar{y}_{N-1}A)$$

and the following conditions are equivalent

1. $\bar{y} < \infty$
2. $\lim_{N \rightarrow \infty} \bar{y}_N < \infty$
3. There exists $c > 0$ such that $P(B + cA \leq c) = 1$.
4. For all $k \in \mathbb{N}$ equality $P(Y_k > 0, A_1 \dots A_k > 1) = 0$ holds.

Proof. We begin by showing equality (4.1). Denote $\text{ess sup}(B + \bar{y}_{N-1}A) = \kappa$ and define function $g: [0, \infty) \rightarrow \mathbb{R} \cup \{\infty\}$ by formula

$$g(c) = \text{ess sup}(B + Ac).$$

We recall from section 2.1 that $\bar{Y}_N \stackrel{d}{=} V_N$, where

$$V_0 = 0, \quad V_N \stackrel{d}{=} B_N + A_N \max(0, V_{N-1}).$$

Because $\text{ess sup } V_N = \text{ess sup } \bar{Y}_N$ and the risks of different years are independent, we get

$$\begin{aligned}\bar{y}_N &= \text{ess sup } \bar{Y}_N \\ &= \text{ess sup } V_N \\ &= \text{ess sup } (B_N + A_N V_{N-1}) \\ &\leq \text{ess sup } g(V_{N-1}) \\ &\leq g(\bar{y}_{N-1}).\end{aligned}$$

This yields $\bar{y}_N \leq \kappa$.

For the remaining direction, assume first $\kappa < \infty$. Fix $\varepsilon > 0$. Now there exists a set $H_1 \in \Omega$ for which $P(H_1) > 0$ and

$$B(\omega) + \bar{y}_{N-1}A(\omega) > \kappa - \varepsilon, \quad \forall \omega \in H_1.$$

We may again choose a number $\eta = \eta_\varepsilon > 0$ in a way that there is a set $H_2 \subset H_1$, where $P(H_2) > 0$, and

$$B(\omega) + (\bar{y}_{N-1} - \eta)A(\omega) > \kappa - \varepsilon, \quad \forall \omega \in H_2.$$

Using the definition of supremum and the independence structure of the process (Y_k) we can find a third set $H_3 \subset H_2$, where $P(H_3) > 0$, and

$$B(\omega) + \bar{Y}_{N-1}(\omega)A(\omega) > \kappa - \varepsilon, \quad \forall \omega \in H_3.$$

This implies $\kappa - \varepsilon \leq \bar{y}_N$ by using the connections between variables \bar{Y}_{N-1} ja V_{N-1} . Equation (4.1) then follows by letting $\varepsilon \rightarrow 0$. Suppose then that $\kappa = \infty$. The above deduction can now be done again by replacing $\kappa - \varepsilon$ with an arbitrarily large number M . This gives the remaining result.

We proceed to the proof of equivalences 1-4. Equivalence of 1 and 2 is clear, since (\bar{y}_N) is a non decreasing sequence of numbers whose limit is \bar{y} .

Assume 2. Denote $\lim_{N \rightarrow \infty} \bar{y}_N = c$. By assumption iv inequality $c > 0$ holds. Using connection

$$\bar{y}_{N+1} = \text{ess sup}(B + \bar{y}_N A)$$

we may deduce that

$$c = \text{ess sup}(B + cA).$$

Hence almost surely

$$B + cA \leq c$$

and 3 is valid.

Assume 3. Set $a_- = \text{ess inf } A$, where the essential infimum is defined analogously to the essential supremum. By assumptions i and iii we see that $a_- \in [0, 1)$. Now, almost surely

$$B \leq c(1 - A) \leq c(1 - a_-).$$

Especially $\bar{y}_1 = \text{ess sup } B \leq c(1 - a_-)$. Therefore

$$\begin{aligned}
\bar{y}_2 &= \text{ess sup}(B + \bar{y}_1 A) \\
&\leq \text{ess sup}(B + c(1 - a_-)A) \\
&= \text{ess sup}(B + cA - ca_-A) \\
&\leq c + \text{ess sup}(-ca_-A) \\
&= c - c(a_-)^2 = c(1 - (a_-)^2).
\end{aligned}$$

In general

$$\bar{y}_N \leq c(1 - (a_-)^N) \rightarrow c < \infty,$$

holds when $N \rightarrow \infty$. This proves 2.

Equivalence of 1 and 4 is precisely the content of theorem first proved by Nyrhinen in [17], Theorem 2. \square

Conditions 3 and 4 of Theorem 1 can be regarded as microscopic and macroscopic ways to see when the essential supremum of the process (Y_k) is unlimited. The first of these tells us what the random vectors (A, B) must satisfy. The latter condition looks at the process (Y_k) on a large scale. Condition 3 is an operational tool that can be used in proofs, whereas condition 4 is difficult to verify in practice.

5 The case of bounded interval

We will begin with a simple observation based on Corollary 1.

Lemma 8. Suppose $N \in \mathbb{N}$. Then

$$(5.1) \quad \mathbb{I}(\bar{Y}_N) = \min_{1 \leq k \leq N} \{ \min_{\theta \in \Theta_k} \mathbb{I}(Y_k | \theta) \},$$

where

$$(5.2) \quad \Theta_k = \{S \subset \Omega : S = \cap_{i=1}^k K_i, \text{ where } K_i = \{B_i > 0\} \text{ or } K_i = \{B_i \leq 0\}\}.$$

Proof. Using Lemma 3 it is clear that $\mathbb{I}(\bar{Y}_N) = \min_{1 \leq k \leq N} \mathbb{I}(Y_k)$. For a fixed k the family Θ_k of equation (5.2) forms a partition of the state space Ω . Hence, by using Corollary 1, equation (5.1) holds. \square

In general, when dependence between the random variables A and B is possible, it is not clear in which set the minimum (5.1) is obtained. Especially interesting interaction between A and B can be observed if the insurance risk partially cancels the financial risk.

To understand this phenomenon, we define the function $h: [0, \infty] \rightarrow [0, \infty]$ by formula

$$(5.3) \quad h(c) = \mathbb{I}(B + cA),$$

when $0 \leq c < \infty$. Because the estimate $B + c_1 A \leq B + c_2 A$ holds for each $0 \leq c_1 < c_2$, we get $\mathbb{I}(B + c_1 A) \geq \mathbb{I}(B + c_2 A)$ and deduce that h is a decreasing function. In addition, h is limited from below by 0. Hence, we can define

$$h(\infty) = \lim_{c \rightarrow \infty} h(c).$$

It turns out the function h of formula (5.3) offers enough information from the random vector (A, B) in order to express the quantities $\mathbb{I}(\bar{Y}_k)$. The following two examples demonstrate that the function h is not necessarily neither right nor left continuous.¹

Example 3. Assume the setting of example 2. Define

$$B = -W_\alpha \mathbf{1}(K = 0) - 2W_\beta \mathbf{1}(K = 1) + 10$$

and

$$A = W_\alpha \mathbf{1}(K = 0) + W_\beta \mathbf{1}(K = 1).$$

Now

$$\mathbb{I}(B + cA) = \begin{cases} \infty & : c \in [0, 1] \\ \alpha & : c \in (1, 2] \\ \beta & : c > 2. \end{cases}$$

Assuming B is a sum of suitable dependent random variables enables us to see the following.

Example 4. Suppose W_γ is defined as in example 2 and $1 < \beta < \alpha$. Set $A = W_{\beta/2} + W_\alpha$ and $B = -W_{\beta/2} + W_{\beta/2}^{1/2} + 1$, where $W_{\beta/2}$ and W_α are independent. Noting that $(c - 1)W_{\beta/2} + W_{\beta/2}^{1/2}$ is bounded for $c \in (0, 1)$ and using results of Lemmas 2 and 4 we obtain

$$\mathbb{I}(B + cA) = \begin{cases} \alpha & : c \in (0, 1) \\ \beta & : c = 1. \end{cases}$$

We may now give a range in which always $\mathbb{I}(\bar{Y}_k)$ is.

Theorem 2. Consider the process (Y_k) of (1.1) consisting of IID vectors (A, B) , where $A > 0$. Then, for any fixed $n \in \mathbb{N}$,

$$(5.4) \quad \mathbb{I}(\bar{Y}_N) \in \left[h(\bar{y}_{N-1}), \lim_{c \rightarrow \bar{y}_{N-1}^-} h(c) \right]$$

where h is the function defined in (5.3).

¹The fact that h is not necessarily left continuous was kindly pointed out by Harri Nyrhinen in a personal communication. Essentially, this idea is formulated in example 4.

Proof. To prove (5.4) we note that for any $\varepsilon > 0$

$$\begin{aligned}
E((\bar{Y}_N)^s \mathbf{1}(\bar{Y}_N > 0)) &= E(((B + A\bar{Y}_{N-1}^+)^+)^s) \\
&\geq E(((B + A\bar{Y}_{N-1}^+)^+)^s \mathbf{1}(\bar{Y}_{N-1} > \bar{y}_{N-1} - \varepsilon)) \\
&\geq E(((B + A(\bar{y}_{N-1} - \varepsilon))^+)^s \mathbf{1}(\bar{Y}_{N-1} > \bar{y}_{N-1} - \varepsilon)) \\
&= E(((B + A(\bar{y}_{N-1} - \varepsilon))^+)^s) P(\bar{Y}_{N-1} > \bar{y}_{N-1} - \varepsilon),
\end{aligned}$$

which is why $E(((B + A(\bar{y}_{N-1} - \varepsilon))^+)^s)$ cannot be infinite when $s < \mathbb{I}(\bar{Y}_N)$. Hence

$$(5.5) \quad \mathbb{I}(\bar{Y}_N) \leq \lim_{c \rightarrow \bar{y}_{N-1}^-} h(c).$$

On the other hand

$$\begin{aligned}
E((\bar{Y}_N)^s \mathbf{1}(\bar{Y}_N > 0)) &= E(((B + A\bar{Y}_{N-1}^+)^+)^s) \\
&\leq E(((B + A\bar{y}_{N-1})^+)^s),
\end{aligned}$$

so

$$(5.6) \quad \mathbb{I}(\bar{Y}_N) \geq h(\bar{y}_{N-1}).$$

Equations (5.5) and (5.6) prove the claim. \square

Corollary 2. *Assume that*

1. *A and B are independent or*
2. *B is non-negative.*

Then

$$\mathbb{I}(\bar{Y}_N) = \min(\mathbb{I}(A), \mathbb{I}(B))$$

for any $N \geq 2$.

Proof. By Lemma 4 and Lemma 2 part 1 the function h of (5.3) is the constant function $\min(\mathbb{I}(A), \mathbb{I}(B))$ for all $c > 0$. \square

What is new in Theorem 2 compared to Theorem 4.1 of [21] is that the risks of the same year may have arbitrary dependence structure. This is possible due to the refined use of moment indices. Corollary also 2 extends the result of [21] beyond independence, if B is non-negative.

6 The case of unbounded interval

This section is dedicated to the proof of the last main theorem. The proof is divided into two parts. The aim of the first part is to obtain an upper bound for the left hand side of (6.1) in the most general possible setting. The second part is divided into different cases.

Theorem 3. Assume that (A_i) and (B_i) satisfy the assumptions I-III of section 1.2. Assume further that $\bar{y} = \infty$. Then

$$(6.1) \quad \limsup_{U_0 \rightarrow \infty} \frac{\log P(T_{U_0} < \infty)}{\log U_0} = -\min(\mathbb{I}^1(A), \mathbb{I}(B)).$$

If Y_∞ of (1.4) is well defined, we set $y_\infty = \text{ess sup } Y_\infty$ and get the following corollary. Equivalent conditions for $y_\infty = \infty$ can be found from [17], Theorem 2.2.

Corollary 3. Suppose I-III of section 1.2 hold and $y_\infty = \infty$. Then

$$(6.2) \quad \limsup_{U_0 \rightarrow \infty} \frac{\log P(Y_\infty > U_0)}{\log U_0} = -\min(\mathbb{I}^1(A), \mathbb{I}(B)).$$

Proof. We see that the assumptions of Theorem 3 are valid. In addition, $P(Y_\infty > 0) > 0$ holds. Now (1.16) yields the result. \square

If B can take values of both signs with positive probability and conditions of Corollary 3 hold, we immediately obtain

$$\mathbb{I}(|Y_\infty|) = \min(\mathbb{I}(-Y_\infty), \mathbb{I}(Y_\infty)) = \min(\mathbb{I}^1(A), \mathbb{I}(|B|)).$$

This is consistent with the Theorem 1.4 of [1].

6.1 Upper bound

Proof. Only the case $0 < s < \min(\mathbb{I}^1(A), \mathbb{I}(B))$ has mathematical content. We begin by noting that

$$(6.3) \quad \bar{Y}_n \mathbf{1}(\bar{Y}_n > 0) \leq \sum_{i=1}^n A_1 \dots A_{i-1} B_i^+.$$

We will consider the following two cases.

1. If $1 < s$ we may use (6.3) and the Minkowski's inequality to obtain

$$\begin{aligned} E((\bar{Y}_n)^s \mathbf{1}(\bar{Y}_n > 0))^{1/s} &\leq \sum_{i=1}^n E((A_1 \dots A_{i-1} B_i^+)^s)^{1/s} \\ &= E((B^+)^s) \sum_{i=1}^n E(A^s)^{(i-1)/s} \\ &\leq E((B^+)^s) \sum_{i=1}^{\infty} E(A^s)^{(i-1)/s} < \infty. \end{aligned}$$

Now, by monotone convergence

$$E(\bar{Y}^s \mathbf{1}(\bar{Y} > 0)) = \lim_{n \rightarrow \infty} E((\bar{Y}_n)^s \mathbf{1}(\bar{Y}_n > 0)) < \infty.$$

2. If $0 < s \leq 1$ we may use the subadditive property of the function $x \mapsto x^s$ for $x \geq 0$ and obtain similarly that $E((\bar{Y}_n)^s \mathbf{1}(\bar{Y} > 0)) < C$ for a constant C that doesn't depend on n . This gives $E(\bar{Y}^s \mathbf{1}(\bar{Y} > 0)) < \infty$ as before.

In either case

$$E(\bar{Y}^s \mathbf{1}(\bar{Y} > 0)) \geq E(\bar{Y}^s \mathbf{1}(\bar{Y} > U_0)) \geq U_0^s P(\bar{Y} > U_0),$$

which yields

$$\limsup_{U_0 \rightarrow \infty} \frac{\log P(\bar{Y} > U_0)}{\log U_0} \leq -s$$

for all $0 < s < \min(\mathbb{I}(A), \mathbb{I}(B))$. Therefore

$$\limsup_{U_0 \rightarrow \infty} \frac{\log P(T_{U_0} < \infty)}{\log U_0} = \limsup_{U_0 \rightarrow \infty} \frac{\log P(\bar{Y} > U_0)}{\log U_0} \leq -\min(\mathbb{I}(A), \mathbb{I}(B)).$$

□

6.2 Lower bound

We begin the proof of the lower bound by restricting ourselves to the most difficult situation. The other cases can then be deduced from this result using estimation methods developed in section 3.

6.2.1 Restricted setting

We assume that

$$(6.4) \quad \min(\mathbb{I}^1(A), \mathbb{I}(B)) > 0 \text{ and } \mathbb{I}^1(A) < \mathbb{I}(A).$$

We will need two additional lemmas. Lemma 9 is merely an observation, but Lemma 10 presents a fundamental property of the process (Y_k) .

Lemma 9. Let χ be a real valued random variable and ξ a strictly positive random variable. Then, for any $c > 0$,

$$(6.5) \quad \begin{aligned} \{\chi + \xi c > c\} &= \{\chi > 0, \xi \geq 1\} \\ &\cup \left\{ \frac{\chi}{1-\xi} < c, \chi \leq 0, \xi > 1 \right\} \\ &\cup \left\{ \frac{\chi}{1-\xi} > c, \chi > 0, \xi < 1 \right\}, \end{aligned}$$

where the sets of the union on the right hand side are separate. Furthermore

$$\lim_{c \rightarrow 0+} \mathbf{1}(\chi + \xi c > c)(\omega) = \mathbf{1}(\chi > 0)(\omega) + \mathbf{1}(\chi = 0, \xi > 1)(\omega)$$

and

$$\lim_{c \rightarrow \infty} \mathbf{1}(\chi + \xi c > c)(\omega) = \mathbf{1}(\xi > 1)(\omega) + \mathbf{1}(\chi > 0, \xi = 1)(\omega)$$

for all $\omega \in \Omega$.

Lemma 10. Assume $\mathbb{I}^1(A) < s < \mathbb{I}(A)$.

Then we may choose $c = c_s$ and $n = n_s$ so that

$$(6.6) \quad E((A_1 \dots A_n)^s \mathbf{1}(Y_n + A_1 \dots A_n c > c)) > 1.$$

Proof. By independence of financial risks

$$(6.7) \quad E((A_1 \dots A_k)^s) = E(A_1^s) \dots E(A_k^s) \rightarrow \infty,$$

as $k \rightarrow \infty$. On the other hand

$$(6.8) \quad E((A_1 \dots A_k)^s) = E((A_1 \dots A_k)^s \mathbf{1}(A_1 \dots A_k > 1)) + E((A_1 \dots A_k)^s \mathbf{1}(A_1 \dots A_k \leq 1)),$$

where

$$E((A_1 \dots A_k)^s \mathbf{1}(A_1 \dots A_k \leq 1)) \leq 1.$$

Since (6.7) grows without limitation while the last term of (6.8) remains uniformly bounded for each k , we see that there is a number $n \in \mathbb{N}$ for which

$$(6.9) \quad E((A_1 \dots A_n)^s \mathbf{1}(A_1 \dots A_n > 1)) > 1.$$

Choose $\chi = Y_n$ and $\xi = A_1 \dots A_n$ in Lemma 9. By representation (6.5) and the positivity of the random variable $(A_1 \dots A_n)^s$ we get for each $c > 0$ the estimate

$$\begin{aligned} & E((A_1 \dots A_n)^s \mathbf{1}(Y_n + A_1 \dots A_n c > c)) \\ & \geq E((A_1 \dots A_n)^s [\mathbf{1}(Y_n > 0, A_1 \dots A_n > 1) + \mathbf{1}(\frac{Y_n}{1 - A_1 \dots A_n} < c, Y_n \leq 0, A_1 \dots A_n > 1)]). \end{aligned}$$

The previous minorant converges to $E((A_1 \dots A_n)^s \mathbf{1}(A_1 \dots A_n > 1))$, as $c \rightarrow \infty$ by monotone convergence theorem. This limit exceeds the level 1 by the result of formula (6.9). \square

We are now ready to present the proof. Firstly, if $\mathbb{I}(B) \leq \mathbb{I}^1(A)$, we may use the trivial bound

$$(6.10) \quad \limsup_{U_0 \rightarrow \infty} \frac{\log P(T_{U_0} < \infty)}{\log U_0} \geq \limsup_{U_0 \rightarrow \infty} \frac{\log P(Y_1 > U_0)}{\log U_0} = -\mathbb{I}(B).$$

Assume then that inequality $\mathbb{I}(B) > \mathbb{I}^1(A)$ holds. Iteration of the stochastic fixed point equation $\bar{Y} \stackrel{d}{=} B + A\bar{Y}^+$ combined with the monotonicity of the function $x \mapsto x^+$ yields

$$\begin{aligned} (\bar{Y})^+ & \stackrel{d}{=} (B_1 + A_1 \bar{Y}^+)^+ \\ & \geq (B_1 + A_1 \bar{Y})^+ \\ & \stackrel{d}{=} (B_1 + A_1 B_2 + A_1 A_2 \bar{Y}^+)^+ \\ & \geq (B_1 + A_1 B_2 + A_1 A_2 \bar{Y})^+ \\ & \vdots \\ & \stackrel{d}{=} (Y_k + A_1 \dots A_k \bar{Y}^+)^+ \\ & \geq (Y_k + A_1 \dots A_k \bar{Y})^+ \end{aligned}$$

for each $k \in \mathbb{N}$. Here the random variable \bar{Y} on the right hand side is independent of the random vectors $(A_1, B_1), \dots, (A_k, B_k)$.

Using the previous estimate we obtain for all $c > 0$ and $k \in \mathbb{N}$

$$\begin{aligned} E((\bar{Y} - c)^s \mathbf{1}(\bar{Y} > c)) &\geq E((Y_k + A_1 \dots A_k \bar{Y} - c)^s \mathbf{1}(Y_k + A_1 \dots A_k \bar{Y} > c)) \\ &\geq E((Y_k + A_1 \dots A_k \bar{Y} - c)^s \mathbf{1}(Y_k + A_1 \dots A_k c > c) \mathbf{1}(\bar{Y} > c)) \\ &\geq E((A_1 \dots A_k (\bar{Y} - c))^s \mathbf{1}(Y_k + A_1 \dots A_k c > c) \mathbf{1}(\bar{Y} > c)) \\ &= E((A_1 \dots A_k)^s \mathbf{1}(Y_k + A_1 \dots A_k c > c)) E((\bar{Y} - c)^s \mathbf{1}(\bar{Y} > c)). \end{aligned}$$

Keeping in mind the assumption $\bar{y} = \infty$ this implies for any $s < \mathbb{I}(\bar{Y})$ that

$$(6.11) \quad E((A_1 \dots A_k)^s \mathbf{1}(Y_k + A_1 \dots A_k c > c)) \leq 1.$$

Define, for given $c > 0$ and $k \in \mathbb{N}$,

$$(6.12) \quad A_{c,k} = A_1 \dots A_k \mathbf{1}(Y_k + A_1 \dots A_k c > c).$$

Equation (6.11) implies the inequality

$$(6.13) \quad \mathbb{I}(\bar{Y}) \leq \mathbb{I}^1(A_{c,k}).$$

Assume $\varepsilon > 0$ is given. Choose $\hat{c} = \hat{c}_{\mathbb{I}^1(A) + \varepsilon}$ and $\hat{n} = \hat{n}_{\mathbb{I}^1(A) + \varepsilon}$ according to Lemma 10. Now

$$\mathbb{I}(\bar{Y}) \leq \mathbb{I}^1(A_{\hat{c}, \hat{n}})$$

by equation (6.13). Since

$$E((A_1 \dots A_{\hat{n}})^{\mathbb{I}^1(A) + \varepsilon} \mathbf{1}(Y_n + A_1 \dots A_{\hat{n}} \hat{c} > \hat{c})) > 1,$$

we deduce the inequality

$$\mathbb{I}^1(A_{\hat{c}, \hat{n}}) \leq \mathbb{I}^1(A) + \varepsilon,$$

which proves (6.1) under assumptions (6.4). \square

6.2.2 General setting

We will now omit assumptions of (6.4) and prove (6.1). If $\mathbb{I}(B) \leq \mathbb{I}^1(A)$, the formula (6.1) holds because the upper limit of chapter 6.1 is always valid and lower bound can be obtained from trivial estimate (6.10).

We may now assume $\mathbb{I}(B) > \mathbb{I}^1(A)$. Fix small numbers $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Then construct random variables A_{ε_1} and B_{ε_2} as in lemmas 6 and 7. Now $A \geq A_{\varepsilon_1}$ and $B \geq B_{\varepsilon_2}$ almost surely. By (3.2) and the solution of the restricted case in 6.2.1 we obtain

$$(6.14) \quad \limsup_{U_0 \rightarrow \infty} \frac{\log P(T_{U_0} < \infty)}{\log U_0} \geq -\min(\mathbb{I}^1(A_{\varepsilon_1}), \mathbb{I}(B_{\varepsilon_2})).$$

Since $\mathbb{I}^1(A_{\varepsilon_1}) = \mathbb{I}^1(A) + \varepsilon_1$ and $\mathbb{I}(B_{\varepsilon_2}) = \mathbb{I}(B) + \varepsilon_2$ and (6.14) holds for arbitrarily small numbers ε_1 and ε_2 , we get

$$\limsup_{U_0 \rightarrow \infty} \frac{\log P(T_{U_0} < \infty)}{\log U_0} = -\min(\mathbb{I}^1(A), \mathbb{I}(B))$$

in all possible cases.

7 Conclusions and comments

7.1 Classical random walk

In section 1 the classical random walk was briefly mentioned. Formally, a random walk can be recovered from model (1.1) by setting $A \equiv 1$. This leads to a process (S_n) , where

$$(7.1) \quad S_n = B_1 + \dots + B_n.$$

Denote $\bar{S} = \sup_n B_n$. The model (1.1) with non-degenerate financial risks converges, whereas the process S_n has a drift. In order for the analysis of the random variable \bar{S} to be meaningful, this drift needs to be negative.

If $E(|B|) < \infty$, the negative drift is known to be equivalent with the condition $E(B) < 0$. Under this assumption Borokov, see [4] page 140 formula (54), derived a representation for the moment index $\mathbb{I}(\bar{S})$ using ascending ladder heights associated with the random process (S_n) . In this case

$$\mathbb{I}(\bar{S}) = \mathbb{I}(B) - 1,$$

which is completely different than the result obtained for the process (Y_n) .

7.2 Final remarks

We want to point out that the quantity (1.13) used in the proof of main theorem has an alternative representation via moment indices.

Let (Z_i) be an IID-sequence of positive random variables. We define

$$(7.2) \quad Z_\infty = \sum_{k=1}^{\infty} Z_1 \dots Z_k$$

and

$$(7.3) \quad \bar{Z} = \sup_k \{Z_1 \dots Z_k\}.$$

The random variable defined in (7.2) makes sense when the stochastic series converges. The following lemma proves the equality of moment indices.

Lemma 11. Let Z be a positive random variable that satisfies assumptions i-iii of Theorem 1. Then the random variables Z_∞ and \bar{Z} , defined by (7.2) and (7.3) respectively, satisfy

$$(7.4) \quad \mathbb{I}^1(Z) = \mathbb{I}(Z_\infty) = \mathbb{I}(\bar{Z}).$$

Proof. To see that $\mathbb{I}^1(Z) = \mathbb{I}(Z_\infty)$, take $Z = A$ and $B \equiv 1$ in (6.1).

For the last equality of (7.4), note first that $\bar{Z} \leq Z_\infty$. By Lemma 2 part 3, $\mathbb{I}(\bar{Z}) \geq \mathbb{I}(Z_\infty)$. Furthermore, for any $k \in \mathbb{N}$ and $s > 0$

$$(7.5) \quad E(\bar{Z}^s) \geq E((Z_1 \dots Z_k)^s) = E(Z^s)^k.$$

Since formula (7.5) renders the inequality $\mathbb{I}^1(Z) < \mathbb{I}(\bar{Z})$ impossible, we conclude that $\mathbb{I}^1(Z) \geq \mathbb{I}(\bar{Z})$. \square

Acknowledgements

Financial support from Finnish doctoral programme in stochastics and statistics (FDPSS) is acknowledged as well as the support from the Centre of Excellence in Computational Inference (COIN). Special thanks are due to Ph.D. Harri Nyrhinen for his diligent guidance throughout the writing process of the paper.

References

- [1] Alsmeyer, G., Iksanov, A., Rösler, U.: On distributional properties of perpetuities. *Journal of Theoretical Probability* **22**(3), 666–682 (2009)
- [2] Baltrūnas, A., Daley, D., Klüppelberg, C.: Tail behaviour of the busy period of a $gi/gi/1$ queue with subexponential service times. *Stochastic processes and their applications* **111**(2), 237–258 (2004)
- [3] Bingham, N., Goldie, C., Teugels, J.: Regular variation, vol. 27. Cambridge University Press (1989)
- [4] Borovkov, A.: Stochastic processes in queueing theory: Applications of mathematics. Springer-Verlag (1976)
- [5] Borovkov, A., Borovkov, K.: Asymptotic analysis of random walks: heavy-tailed distributions (encyclopedia of mathematics & its applications, n 118). *Recherche* **67**, 02 (2008)
- [6] Collamore, J.: Random recurrence equations and ruin in a markov-dependent stochastic economic environment. *The Annals of Applied Probability* **19**(4), 1404–1458 (2009)
- [7] Daley, D.: The moment index of minima. *Journal of Applied Probability* pp. 33–36 (2001)
- [8] Daley, D., Goldie, C.: The moment index of minima (ii). *Statistics & probability letters* **76**(8), 831–837 (2006)
- [9] Dembo, A., Zeitouni, O.: Large deviations techniques and applications, vol. 38. Springer Verlag (2009)
- [10] Embrechts, P., Klüppelberg, C., Mikosch, T.: Modelling extremal events for insurance and finance, vol. 33. Springer Verlag (1997)
- [11] Foss, S., Korshunov, D., Zachary, S.: An introduction to heavy-tailed and subexponential distributions. Springer Verlag (2011)
- [12] Goldie, C.: Implicit renewal theory and tails of solutions of random equations. *Annals of Applied Probability* pp. 126–166 (1991)

- [13] Goldie, C., Maller, R.: Stability of perpetuities. *Annals of probability* pp. 1195–1218 (2000)
- [14] Nummelin, E.: General irreducible Markov chains and non-negative operators. *Cambridge Tracts in Mathematics*, 83 (1984)
- [15] Nyrhinen, H.: On the ruin probabilities in a general economic environment. *Stochastic Processes and their Applications* **83**(2), 319 – 330 (1999)
- [16] Nyrhinen, H.: Finite and infinite time ruin probabilities in a stochastic economic environment. *Stochastic Processes and their Applications* **92**(2), 265 – 285 (2001)
- [17] Nyrhinen, H.: On stochastic difference equations in insurance ruin theory. *Journal of Difference Equations and Applications* **18**(8), 1345–1353 (2012)
- [18] Paulsen, J.: Ruin theory with compounding assets - a survey. *Insurance: Mathematics and Economics* **22**(1), 3–16 (1998)
- [19] Paulsen, J.: On cramer-like asymptotics for risk processes with stochastic return on investments. *Annals of Applied Probability* **12**(4), 1247–1260 (2002)
- [20] Rockafellar, R.: *Convex analysis*, vol. 28. Princeton Univ Pr (1997)
- [21] Tang, Q., Tsitsiashvili, G.: Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. *Stochastic Processes and their Applications* **108**(2), 299 – 325 (2003)
- [22] Tang, Q., Tsitsiashvili, G.: Randomly weighted sums of subexponential random variables with application to ruin theory. *Extremes* **6**, 171–188 (2003)
- [23] Tang, Q., Tsitsiashvili, G.: Finite-and infinite-time ruin probabilities in the presence of stochastic returns on investments. *Advances in Applied Probability* **36**(4), 1278–1299 (2004)